UNIVERSITY OF PISA
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PhD Course in Structural Engineering – XVII Cycle

The Saint Venant problem: an approach based on the average displacements values over the cross-sections

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1. SUMMARY

In the present work, the solution of the Saint Venant problem is attained in a different way from the classic method, since the proposed approach relies from the beginning on the average displacements and rotations of the cylinder cross-section. Basically, it is emphasised the role of such entities, which are commonly involved only in the classic Beam Theory.

The solution of the problem is developed in a general way, considering the indefinite equilibrium equations, the compatibility equations and the constitutive model typical of the elastic continuum mechanics; than the simplification of the stress tensor is applied, and the boundary conditions are imposed. As a consequence of the basic hypothesis, the expression of the local displacements field involves the average displacements and rotations of the cross-section; in addition, it contains the characteristic deformations of the solid infinitesimal transversal portion.

Moving from such expressions, the relations between deformations and internal forces are derived; these relations, together with the beam equilibrium equations, enable us to determine all the parameters which characterise the continuum deformation status.

Operating in the abovementioned way, the problem which must be faced at first is to individuate the most suitable system of kinematic quantities, to represent the motion of the cross-section. In effect, while the section torsional rotation is univocally definable, due to the rigid-body constraint affecting the cylinder, the same condition is not occurring for the two flexural rotations, whenever a section warping is present. In this situation, the definition of the characteristic displacements and rotations is not univocal; in addition, the solution of the problem in terms of displacements assumes different expressions depending on the choice made with regards to such definitions.

In the present study, we refer to two different expressions of the generalised section displacements, the first one is obtained through geometrical assessments, the second one is related to energetic principles. It is therefore pointed out how the expressions of the cylinder motion depend on the abovementioned settings, and especially how, in the two cases, the shear deformation factors and the position of the torsional centre of the section prove to be different.

2. DISPALCEMENTS EXPRESSION

2.1. Average displacements and strains over the section

Consider a continuum with cylindrical shape, featured by a main longitudinal axis z and a cross section A with boundary Γ, constituted by a linear elastic, homogeneous, isotropic material, subjected to the load and restraint conditions defined by the Saint Venant problem. Superimposing the transversal coordinate axes x and y with the principal axes of inertia of section A (fig. 1), the local displacement field in the section may be put in the following form:

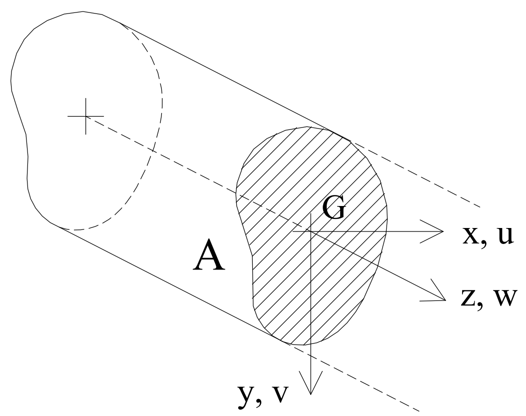

Fig. 1. The Saint Venant cylinder.

$$
\begin{aligned}
u &= u_m + \left.\frac{\partial u}{\partial x}\right|_m x + \left.\frac{\partial u}{\partial y}\right|_m y + u^* , \\
v &= v_m + \left.\frac{\partial v}{\partial x}\right|_m x + \left.\frac{\partial v}{\partial y}\right|_m y + v^* , \qquad (2.1) \\
w &= w_m + \left.\frac{\partial w}{\partial x}\right|_m x + \left.\frac{\partial w}{\partial y}\right|_m y + w^* .
\end{aligned}
$$

The terms u_m, v_m, w_m denote the average displacements of the section along x, y, and z directions, defined as

$$
u_m = \frac{1}{A}\int_A u\, dA \quad , \quad v_m = \frac{1}{A}\int_A v\, dA \quad , \quad w_m = \frac{1}{A}\int_A w\, dA ,
$$

while the quantities $\left.\frac{\partial u_i}{\partial x_j}\right|_m$ represent the averages over the section of the displacements partial derivatives with respect to x and y directions, defined as

$$\left.\frac{\partial u_i}{\partial x_j}\right|_m = \frac{1}{A}\int_A \frac{\partial u_i}{\partial x_j} dA .$$

On the basis of these assumptions, functions u^*, v^*, and w^* describe a displacement field featuring null average values and null average partial derivatives in the section.

It is convenient to introduce this kind of quantities in the expression of the displacement field, since they easily relate to the mean rotations and strains over the section, intended as the integral means of rotations and strains affecting the differential elements. In effect, from the decomposition of the displacement gradient tensor, it emerges that

$$\varepsilon_x = \frac{\partial u}{\partial x} \quad , \quad \varepsilon_y = \frac{\partial v}{\partial y}$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad , \quad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} ,$$

and at the same time

$$\frac{\partial u}{\partial y} = \frac{\gamma_{xy}}{2} - \omega_z \quad , \quad \frac{\partial v}{\partial x} = \frac{\gamma_{xy}}{2} + \omega_z ,$$

from which, referring to the average values, we obtain

$$\left.\frac{\partial u}{\partial x}\right|_m = \frac{1}{A}\int_A \varepsilon_x \, dA = \varepsilon_{xm} \quad , \quad \left.\frac{\partial u}{\partial y}\right|_m = \frac{1}{A}\int_A \left(\frac{\gamma_{xy}}{2} - \omega_z\right) dA = \frac{\gamma_{xym}}{2} - \omega_{zm}$$

$$\left.\frac{\partial v}{\partial x}\right|_m = \frac{1}{A}\int_A \left(\frac{\gamma_{xy}}{2} + \omega_z\right) dA = \frac{\gamma_{xym}}{2} + \omega_{zm} \quad , \quad \left.\frac{\partial v}{\partial y}\right|_m = \frac{1}{A}\int_A \varepsilon_y \, dA = \varepsilon_{ym}$$

Considering again equations (2.1), the quantities $-\left.\frac{\partial w}{\partial x}\right|_m$, $\left.\frac{\partial w}{\partial y}\right|_m$ represent the average rotation over area A of the differential surface elements with normal vector in z direction, respectively around y and x axes; they can therefore be regarded as the average rotations of the section around such axes, and they will be denoted by the following terms

$$\varphi_x = \left.\frac{\partial w}{\partial y}\right|_m = \frac{1}{A}\int_A \frac{\partial w}{\partial y} dA ,$$
$$\varphi_y = -\left.\frac{\partial w}{\partial x}\right|_m = -\frac{1}{A}\int_A \frac{\partial w}{\partial x} dA . \qquad (2.2)$$

Considering again the strain tensor, we find

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \quad , \quad \gamma_{zy} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \, ,$$

for which, in terms of average values, the following relations are obtained

$$\begin{aligned} \varphi_x &= \gamma_{zym} - \frac{dv_m}{dz} \, , \\ \varphi_y &= -\gamma_{zxm} + \frac{du_m}{dz} \, . \end{aligned} \qquad (2.3)$$

Assumed to identify the mean displacements of the section with the displacements of the cylinder axis, relations (2.3) show the dependence between the average rotation of the section and the axis rotation, when the infinitesimal portion of the solid is affected by a non-zero mean angular strain.

Substituting the derived expressions in (2.1), it results that the displacement field may be written in the following form

$$\begin{aligned} u &= u_m + \varepsilon_{xm} \, x + \left(\frac{\gamma_{xym}}{2} - \omega_{zm} \right) y + u^* \, , \\ v &= v_m + \left(\frac{\gamma_{xym}}{2} + \omega_{zm} \right) x + \varepsilon_{ym} \, y + v^* \, , \\ w &= w_m - \varphi_y \, x + \varphi_x \, y + w^* \, , \end{aligned} \qquad (2.4)$$

It involves, in an absolutely general way, the average displacements and rotations of the section, besides the null average displacement functions u^*, v^*, and w^*.

On the basis of the Saint Venant simplifying hypothesis with regards to the stress tensor, local strains must comply, in each point of the section, with the following relations:

$$\begin{aligned} \gamma_{xy} &= 0 \, , \\ \varepsilon_x &= \varepsilon_y = -\nu \, \varepsilon_z \, , \end{aligned} \qquad (2.5)$$

in the same way, in terms of average values over the section, we find

$$\begin{aligned} \gamma_{xym} &= 0 \, , \\ \varepsilon_{xm} &= \varepsilon_{ym} = -\nu \, \varepsilon_{zm} \, , \end{aligned}$$

having denoted the mean axial strain in z direction as ε_{zm}. Substituting the obtained values in equations (2.4), the displacements expressions typical of the Saint Venant problem are derived

$$\begin{aligned} u &= u_m - \nu\,\varepsilon_{zm}\,x - \omega_{zm}\,y + u^*\,, \\ v &= v_m + \omega_{zm}\,x - \nu\,\varepsilon_{zm}\,y + v^*\,, \\ w &= w_m - \varphi_y\,x + \varphi_x\,y + w^*\,. \end{aligned} \qquad (2.6)$$

The differentiation of expressions (2.6) yields the strains affecting the infinitesimal volume element, which result

$$\begin{aligned} \varepsilon_x &= -\nu\,\varepsilon_{zm} + \frac{\partial u^*}{\partial x}\,, & \gamma_{zx} &= \gamma_{zxm} - \theta\,y + \frac{\partial w^*}{\partial x} + \frac{\partial u^*}{\partial z}\,, \\ \varepsilon_y &= -\nu\,\varepsilon_{zm} + \frac{\partial v^*}{\partial y}\,, & \gamma_{zy} &= \gamma_{zym} + \theta\,x + \frac{\partial w^*}{\partial y} + \frac{\partial v^*}{\partial z}\,, \\ \varepsilon_z &= \varepsilon_{zm} - \chi_y\,x - \chi_x\,y + \frac{\partial w^*}{\partial z}\,, & \gamma_{xy} &= \frac{\partial u^*}{\partial y} + \frac{\partial v^*}{\partial x}\,. \end{aligned} \qquad (2.7)$$

In these expressions, two new kinds of characteristic deformations of the section are pointed out: the flexural curvatures χ_x and χ_y, and the torsional curvature θ, defined as

$$\chi_x = -\frac{d\varphi_x}{dz}\,, \quad \chi_y = \frac{d\varphi_y}{dz}\,, \quad \theta = \frac{d\omega_{zm}}{dz}\,.$$

Essentially, the displacement field and the strain distribution of the continuum are expressed in terms of six displacement parameters, three translations and three rotations, and six deformation parameters. Usually in literature, contrary to the present study, such quantities are not explicitly involved in the solution of the Saint Venant problem, but they are only considered subsequently, as part of the classic Beam Theory.

2.2. Derivation of displacements u*, v* and w*.

Consider the local equilibrium equations, expressed in terms of strains, in the typical form they assume within the Saint Venant theory

$$
\begin{aligned}
&\frac{\partial \gamma_{xz}}{\partial z} = 0\,, \\
&\frac{\partial \gamma_{yz}}{\partial z} = 0\,, \qquad\qquad (2.8) \\
&\frac{\partial \gamma_{zx}}{\partial x} + \frac{\partial \gamma_{zy}}{\partial y} = -2\,(1+\nu)\,\frac{\partial \varepsilon_z}{\partial z}\,.
\end{aligned}
$$

From such relations, the independence of functions γ_z and their mean values from variable z is derived, as well as the linearity of the ε_z distribution over the section. This latter condition implies that the term $\varepsilon_z^* = \partial w^* / \partial z$, in the third of equations (2.7), is identically null, and therefore the displacement w^* is exclusively function of variables x and y.

The local strains expressions (2.7) yield the displacement functions u^* and v^*, by means of punctually imposing conditions (2.5). For these quantities the following expressions are obtained

$$
\begin{aligned}
u^* &= \nu \left[\chi_y \left(\frac{x^2 - y^2}{2} - \frac{J_y - J_x}{2A} \right) + \chi_x xy \right], \\
v^* &= \nu \left[\chi_x \left(\frac{y^2 - x^2}{2} - \frac{J_x - J_y}{2A} \right) + \chi_y xy \right],
\end{aligned}
\qquad (2.9)
$$

provided that x and y axes are coincident with the section principal axes of inertia.

In order to derive the expression of $w^*(x,y)$, consider the third one of equations (2.8), plus the boundary condition in terms of angular strains

$$
\frac{\partial \gamma_{zx}}{\partial x} + \frac{\partial \gamma_{zy}}{\partial y} = -2\,(1+\nu)\,\frac{\partial \varepsilon_z}{\partial z} \quad \text{in} \quad A\,, \qquad (2.10)
$$

$$
\boldsymbol{\gamma}_z^{\mathbf{T}}\,\mathbf{n} = \begin{bmatrix} \gamma_{zx} \\ \gamma_{zy} \end{bmatrix}^{\mathbf{T}} \mathbf{n} = 0 \quad \text{on} \quad \Gamma\,, \qquad (2.11)
$$

in which the outward normal versor to the section border has been denoted as **n** (fig. 2). In these conditions it results

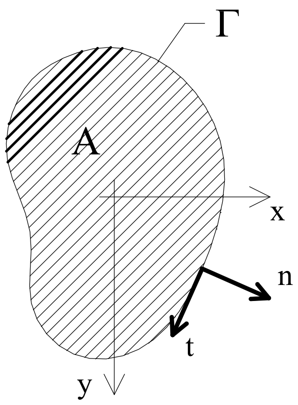

Fig. 2. Tangential and normal vector to the section border.

$$\frac{d\varepsilon_{zm}}{dz} = 0\,, \qquad (2.12)$$

$$\frac{d\chi_x}{dz} = -\frac{A}{2(1+\nu)J_x}\gamma_{zym}\,, \qquad (2.13)$$

$$\frac{d\chi_y}{dz} = -\frac{A}{2(1+\nu)J_y}\gamma_{zxm}\,. \qquad (2.14)$$

The first one is obtained integrating equation (2.10) over section A; on the other hand, multiplying both members of equation (2.10) by the term y, we find

$$\int_A \frac{\partial \gamma_{zx}}{\partial x} y + \frac{\partial \gamma_{zy}}{\partial y} y\, dA = -2(1+\nu)\int_A \frac{\partial \varepsilon_z}{\partial z} y\, dA\,,$$

$$\int_A \frac{\partial}{\partial x}(\gamma_{zx} y) + \frac{\partial}{\partial y}(\gamma_{zy} y) - \gamma_{zy}\, dA = 2(1+\nu)\frac{d\chi_x}{dz}\int_A y^2\, dA\,,$$

$$\int_\Gamma y\,\boldsymbol{\gamma}_z^{\mathbf{T}}\mathbf{n}\, ds - A\gamma_{zym} = 2(1+\nu)J_x\frac{d\chi_x}{dz}\,,$$

from which, on the basis of boundary condition (2.11), equation (2.13) is derived. In the same way, the operation of multiplying by the term x yields equation (2.14).

Substituting expressions (2.9) in equations (2.7), and taking into account conditions (2.13) and (2.14), the angular strains may be written in the following form

$$\gamma_{zx} = \gamma_{zxm}\left[1 + \frac{f(x,y)}{J_y}\right] - \gamma_{zym}\frac{g(x,y)}{J_x} - \theta\, y + \frac{\partial w^*}{\partial x}\,,$$
$$\gamma_{zy} = \gamma_{zym}\left[1 - \frac{f(x,y)}{J_x}\right] - \gamma_{zxm}\frac{g(x,y)}{J_y} + \theta\, x + \frac{\partial w^*}{\partial y}\,, \qquad (2.15)$$

in which it has been put

$$
\begin{aligned}
f(x,y) &= \frac{\nu A}{2(1+\nu)}\left(\frac{y^2-x^2}{2}-\frac{J_x-J_y}{2A}\right), \\
g(x,y) &= \frac{\nu A}{2(1+\nu)}xy\,.
\end{aligned}
\tag{2.16}
$$

At this point, the introduction of local strain expressions (2.15) in equations (2.10) and (2.11) yields the conditions that displacement function $w^*(x,y)$ is required to fulfil:

$$
\Delta w^* = -\frac{A}{1+\nu}\left(\frac{\gamma_{zym}}{J_x}y+\frac{\gamma_{zxm}}{J_y}x\right) \quad \text{in} \quad A,
$$

$$
\mathbf{n}^{\mathrm{T}}\nabla w^* = \frac{\gamma_{zym}}{J_x}\begin{bmatrix} g(x,y) \\ f(x,y)-J_x \end{bmatrix}^{\mathrm{T}}\mathbf{n}+\frac{\gamma_{zxm}}{J_y}\begin{bmatrix} -f(x,y)-J_y \\ g(x,y) \end{bmatrix}^{\mathrm{T}}\mathbf{n}+\theta\begin{bmatrix} y \\ -x \end{bmatrix}^{\mathrm{T}}\mathbf{n}=0 \quad \text{on} \quad \Gamma.
$$

The displacement function $w^*(x,y)$ may be expressed as the sum of three contributions, each one referring to a single deformation among γ_{zxm}, γ_{zym}, and θ:

$$
w^*(x,y)=\gamma_{zxm}\,\psi_{Tx}(x,y)+\gamma_{zym}\,\psi_{Ty}(x,y)+\theta\,\psi_M(x,y)\,.
$$

In this case, the general solution of the problem is obtained solving three independent Dini-Neumann boundary problems, defined by the following relations

$$
\begin{aligned}
&\Delta\psi_M = 0 \quad \text{in} \quad A, && \mathbf{n}^{\mathrm{T}}\nabla\psi_M = \begin{bmatrix} y \\ -x \end{bmatrix}^{\mathrm{T}}\mathbf{n} \quad \text{on} \quad \Gamma, \\
&\Delta\psi_{Ty} = -\frac{A}{(1+\nu)J_x}y \quad \text{in} \quad A, && \mathbf{n}^{\mathrm{T}}\nabla\psi_{Ty} = \frac{1}{J_x}\begin{bmatrix} g(x,y) \\ f(x,y)-J_x \end{bmatrix}^{\mathrm{T}}\mathbf{n} \quad \text{on} \quad \Gamma, \\
&\Delta\psi_{Tx} = -\frac{A}{(1+\nu)J_y}x \quad \text{in} \quad A, && \mathbf{n}^{\mathrm{T}}\nabla\psi_{Tx} = \frac{1}{J_y}\begin{bmatrix} -f(x,y)-J_y \\ g(x,y) \end{bmatrix}^{\mathrm{T}}\mathbf{n} \quad \text{on} \quad \Gamma.
\end{aligned}
\tag{2.17}
$$

Functions ψ_M, ψ_{Ty}, ψ_{Tx}, are univocally determined by the abovementioned boundary problems, and they exclusively depend on the shape of the section; besides, they are defined up to an additive constant, to be determined imposing that their mean value is null throughout the section. In the same way, implicitly to the definition of the displacements distribution $w^*(x,y)$, functions ψ feature null average partial derivatives over the section. Ultimately, they may be regarded as the contributions to the section warping, due to the two shear deformations and to the torsional deformation.

2.3. Displacements, strains and characteristic parameters of the section

On the basis of the issues so far discussed, the generic section of the Saint Venant cylinder undergoes a motion that may be described by the following expressions

$$
\begin{aligned}
u &= u_m - \nu\,\varepsilon_{zm}\,x - \omega_{zm}\,y + \frac{2(1+\nu)}{A}\left[-\chi_y\,f(x,y) + \chi_x\,g(x,y)\right],\\
v &= v_m + \omega_{zm}\,x - \nu\,\varepsilon_{zm}\,y + \frac{2(1+\nu)}{A}\left[\chi_x\,f(x,y) + \chi_y\,g(x,y)\right],\\
w &= w_m - \varphi_y\,x + \varphi_x\,y + \gamma_{zxm}\,\psi_{Tx}(x,y) + \gamma_{zym}\,\psi_{Ty}(x,y) + \theta\,\psi_M(x,y)\,.
\end{aligned}
\tag{2.18}
$$

These ones comprise the average displacements and rotations of the section, the parameters featuring its deformation status, two given functions of variables x and y, whose expression is provided by (2.16), the warping functions ψ, defined by the boundary problems (2.17), to be regarded as known once the shape of the section is established.

On the other hand, the expressions of local strains result

$$
\begin{aligned}
\varepsilon_z &= \varepsilon_{zm} - \chi_y\,x - \chi_x\,y\,,\\
\gamma_{zx} &= \theta\left(\frac{\partial\psi_M}{\partial x} - y\right) + \gamma_{zxm}\left[1 + \frac{\partial\psi_{Tx}}{\partial x} + \frac{f(x,y)}{J_y}\right] + \gamma_{zym}\left[\frac{\partial\psi_{Ty}}{\partial x} - \frac{g(x,y)}{J_x}\right],\\
\gamma_{zy} &= \theta\left(\frac{\partial\psi_M}{\partial y} + x\right) + \gamma_{zym}\left[1 + \frac{\partial\psi_{Ty}}{\partial y} - \frac{f(x,y)}{J_x}\right] + \gamma_{zxm}\left[\frac{\partial\psi_{Tx}}{\partial y} - \frac{g(x,y)}{J_y}\right],
\end{aligned}
\tag{2.19}
$$

which only feature the section deformations parameters, besides the functions f, g, and ψ.

The application of the following definitions of the internal forces

$$
\begin{aligned}
N &= \int_A \sigma_z\,dA\,, & M_x &= \int_A \sigma_z y\,dA\,,\\
T_x &= \int_A \tau_{zx}\,dA\,, & M_y &= -\int_A \sigma_z x\,dA\,,\\
T_y &= \int_A \tau_{zy}\,dA\,, & M_z &= \int_A \left(x\,\tau_{zy} - y\,\tau_{zx}\right) dA\,,
\end{aligned}
$$

yields the existing relations between these quantities and the deformation parameters of the cross-section; these relations, known as constitutive model, result

$$
\begin{aligned}
&N = EA\,\varepsilon_{zm} , && M_x = -EJ_x\,\chi_x , \\
&T_x = GA\,\gamma_{zxm} , && M_y = EJ_y\,\chi_y , \\
&T_y = GA\,\gamma_{zym} , && M_z - T_y\,\delta_{tx} + T_x\,\delta_{ty} = \frac{GJ_0}{q}\theta .
\end{aligned}
\tag{2.20}
$$

Within them, the term J_0 indicates the section polar moment of inertia, while the quantities q and $\boldsymbol{\delta}_t = \left[\delta_{tx}, \delta_{ty}\right]^{\mathrm{T}}$ respectively denote the torsional deformation factor and the position vector of the shear centre, whose expressions are

$$
q = \frac{J_0}{J_0 + \int_A \left(\frac{\partial \psi_M}{\partial y} x - \frac{\partial \psi_M}{\partial x} y \right) dA} = \frac{J_0}{J_0 - \int_A \left[\left(\frac{\partial \psi_M}{\partial x} \right)^2 + \left(\frac{\partial \psi_M}{\partial y} \right)^2 \right] dA} ,
$$

$$
\delta_{tx} = \frac{1}{A} \int_A \left(\frac{\partial \psi_{Ty}}{\partial y} x - \frac{\partial \psi_{Ty}}{\partial x} y \right) dA - \frac{1}{J_x A} \int_A \left[f(x,y)\,x - g(x,y)\,y \right] dA ,
$$

$$
\delta_{ty} = -\frac{1}{A} \int_A \left(\frac{\partial \psi_{Tx}}{\partial y} x - \frac{\partial \psi_{Tx}}{\partial x} y \right) dA - \frac{1}{J_y A} \int_A \left[-g(x,y)\,x + f(x,y)\,y \right] dA .
$$

The evaluation of these quantities only regards the ψ functions, and therefore their values are exclusively dependent on the shape of the section.

The characteristic internal forces, acting on the generic section of the cylinder, are derived imposing the equilibrium conditions; subsequently, the deformations featured by the continuum infinitesimal transversal portion are determined by equations (2.20). Finally, all displacement parameters included in the displacements field expressions (2.18) are obtained integrating the following kinematic compatibility equations

$$
\begin{aligned}
&\varepsilon_{zm} = \frac{dw_m}{dz} , && \gamma_{zxm} = -\varphi_y + \frac{du_m}{dz} , \\
&\chi_x = -\frac{d\varphi_x}{dz} = \frac{d^2 v_m}{dz^2} , && \gamma_{zym} = \varphi_x + \frac{dv_m}{dz} , \\
&\chi_y = \frac{d\varphi_y}{dz} = \frac{d^2 u_m}{dz^2} , && \theta = \frac{d\omega_{zm}}{dz} .
\end{aligned}
$$

These equations define the deformation parameters in terms of derivatives of the generalised displacements. On their basis, we can affirm that the axis inclination of the solid in a single plane exclusively depends on the shear deformation in the plane itself, and in particular, the axis inclination is not affected by the torsional deformation of the cylinder. This means that, in the present model, the torsional centre coincides with the section centroid.

3. SECTION DISPLACEMENTS AND DEFORMATION WORK

3.1. Generalised displacements of the section and external work

Since the expressions of punctual displacements and strains are provided, it is possible to evaluate the deformation work produced within the cylinder; in addition, we set the goal to determine a system of six displacement parameters of the section such that, considered a cylinder elementary portion with differential length *dz*, the external work effected on this portion may be expressed as the differential of the following quantity

$$L_e(z) = N\,w' + M_x\,\varphi_x' + M_y\,\varphi_y' + T_x\,u' + T_y\,v' + M_z\,\omega_z'$$

The external work produced on the portion must hence result

$$dL_e(z) = \left[N\frac{dw'}{dz} + M_x\frac{d\varphi_x'}{dz} + M_y\frac{d\varphi_y'}{dz} + T_x\left(-\varphi_y' + \frac{du'}{dz}\right) + T_y\left(\varphi_x' + \frac{dv'}{dz}\right) + M_z\frac{d\omega_z'}{dz}\right]dz \qquad (3.1)$$

Furthermore, from the expressions of displacements and strains provided by (2.18), the same work may alternatively be evaluated as

$$dL_e(z) = d\int_A \left(\sigma_z w + \tau_{zx} u + \tau_{zy} v\right) dA = \int_A \left(\frac{\partial \sigma_z}{\partial z} w + \sigma_z \varepsilon_z + \tau_{zx}\frac{\partial u}{\partial z} + \tau_{zy}\frac{\partial v}{\partial z}\right) dA\,dz\,.$$

In this special case we have

$$\begin{aligned}
&\int_A \left(\frac{\partial \sigma_z}{\partial z} w + \sigma_z \varepsilon_z\right) dA = N\,\varepsilon_{zm} - M_x\,\chi_x + M_y\,\chi_y + \\
&\qquad + T_x\left(-\varphi_y + \gamma_{zxm}\,\mathrm{X}_x + \gamma_{zym}\,\mathrm{X}_{xy} + \theta\,\delta_{cy}\right) + T_y\left(\varphi_x + \gamma_{zxm}\,\mathrm{X}_{yx} + \gamma_{zym}\,\mathrm{X}_y - \theta\,\delta_{cx}\right), \\
&\int_A \left(\tau_{zx}\frac{\partial u}{\partial z} + \tau_{zy}\frac{\partial v}{\partial z}\right) dA = T_x\frac{du_m}{dz} + T_y\frac{dv_m}{dz} + M_z\,\theta + T_x\,\gamma_{zxm}\,\mathrm{X}_x^* + T_x\,\gamma_{zym}\,\mathrm{X}_{yx}^* + \\
&\qquad + T_y\,\gamma_{zxm}\,\mathrm{X}_{xy}^* + T_y\,\gamma_{zym}\,\mathrm{X}_y^* + \left(M_z + T_x\,\delta_{ty} - T_y\,\delta_{tx}\right)\frac{qA}{J_0}\left(-\gamma_{zxm}\,\delta_{cy}^* + \gamma_{zym}\,\delta_{cx}^*\right)
\end{aligned} \qquad (3.2)$$

in which it has been put

$$\delta_{cx} = -\frac{1}{J_x}\int_A \psi_M \, y \, dA\,, \quad \delta_{cx}^* = -\frac{1}{AJ_x}\int_A \left[\frac{\partial \psi_M}{\partial x} g + \frac{\partial \psi_M}{\partial y} f + \left(f\,x - g\,y\right)\right] dA\,,$$

$$\delta_{cy} = \frac{1}{J_y}\int_A \psi_M \, x \, dA\,, \quad \delta_{cy}^* = -\frac{1}{AJ_y}\int_A \left[\frac{\partial \psi_M}{\partial x} f - \frac{\partial \psi_M}{\partial y} g - \left(f\,y + g\,x\right)\right] dA\,,$$

$$\begin{aligned}
\mathrm{X}_x &= \frac{1}{J_y}\int_A \psi_{Tx} \, x \, dA\,, \quad \mathrm{X}_x^* = \frac{1}{AJ_y}\int_A \left[\frac{\partial \psi_{Tx}}{\partial x} f - \frac{\partial \psi_{Tx}}{\partial y} g + \frac{f^2+g^2}{J_y}\right] dA\,, \\
\mathrm{X}_{xy} &= \frac{1}{J_y}\int_A \psi_{Ty} \, x \, dA\,, \quad \mathrm{X}_{xy}^* = \frac{1}{AJ_y}\int_A \left[\frac{\partial \psi_{Ty}}{\partial x} f - \frac{\partial \psi_{Ty}}{\partial y} g\right] dA\,, \\
\mathrm{X}_y &= \frac{1}{J_x}\int_A \psi_{Ty} \, y \, dA\,, \quad \mathrm{X}_y^* = -\frac{1}{AJ_x}\int_A \left[\frac{\partial \psi_{Ty}}{\partial x} g + \frac{\partial \psi_{Ty}}{\partial y} f - \frac{f^2+g^2}{J_x}\right] dA\,, \\
\mathrm{X}_{yx} &= \frac{1}{J_x}\int_A \psi_{Tx} \, y \, dA\,, \quad \mathrm{X}_{yx}^* = -\frac{1}{AJ_x}\int_A \left[\frac{\partial \psi_{Tx}}{\partial x} g + \frac{\partial \psi_{Tx}}{\partial y} f\right] dA\,.
\end{aligned} \tag{3.3}$$

Summing all the corresponding members relating to the normal strains and to the angular stains, and taking into account relations (2.3), the overall expression of the work turns out to be

$$\begin{aligned}
\frac{dL_e(z)}{dz} &= N\,\varepsilon_{zm} - M_x\,\chi_x + M_y\,\chi_y + \\
&+ T_x\left[\gamma_{zxm}\left(1 + \mathrm{X}_x + \mathrm{X}_x^* - \frac{qA}{J_0}\delta_{ty}\,\delta_{cy}^*\right) + \gamma_{zym}\left(\mathrm{X}_{xy} + \mathrm{X}_{yx}^* + \frac{qA}{J_0}\delta_{ty}\,\delta_{cx}^*\right) + \theta\,\delta_{cy}\right] + \\
&+ T_y\left[\gamma_{zxm}\left(\mathrm{X}_{yx} + \mathrm{X}_{xy}^* + \frac{qA}{J_0}\delta_{tx}\,\delta_{cy}^*\right) + \gamma_{zym}\left(1 + \mathrm{X}_y + \mathrm{X}_y^* - \frac{qA}{J_0}\delta_{tx}\,\delta_{cx}^*\right) - \theta\,\delta_{cx}\right] + \\
&+ M_z\left[\theta + \frac{qA}{J_0}\left(-\gamma_{zxm}\,\delta_{cy}^* + \gamma_{zym}\,\delta_{cx}^*\right)\right].
\end{aligned} \tag{3.4}$$

Whereas, substituting in equations (3.2) the constitutive model (2.20), relating internal forces and deformations, the external work may be rewritten in the following form

$$\begin{aligned}
\frac{dL_e(z)}{dz} &= N\,\varepsilon_{zm} - M_x\,\chi_x + M_y\,\chi_y + T_x\left[\gamma_{zxm}\left(1 + \mathrm{X}_x + \mathrm{X}_x^*\right) + \gamma_{zym}\left(\mathrm{X}_{xy} + \mathrm{X}_{yx}^*\right) + \theta\,\delta_{cy}\right] + \\
&+ T_y\left[\gamma_{zxm}\left(\mathrm{X}_{yx} + \mathrm{X}_{xy}^*\right) + \gamma_{zym}\left(1 + \mathrm{X}_y + \mathrm{X}_y^*\right) - \theta\,\delta_{cx}\right] + M_z\,\theta + \left(-T_x\,\delta_{cy}^* + T_y\,\delta_{cx}^*\right)\theta,
\end{aligned}$$

i.e., summing corresponding terms,

$$\frac{dL_e}{dz} = N\,\varepsilon_{zm} - M_x\,\chi_x + M_y\,\chi_y + T_x\left[\gamma_{zxm}\left(1+\mathrm{X}'_x\right)+\gamma_{zym}\,\mathrm{X}'_{xy}+\theta\,\delta'_{cy}\right]+ \\ +T_y\left[\gamma_{zxm}\,\mathrm{X}'_{yx}+\gamma_{zym}\left(1+\mathrm{X}'_y\right)-\theta\,\delta'_{cx}\right]+M_z\,\theta\,, \tag{3.5}$$

having introduced in the above expression the following quantities

$$\begin{aligned} &\mathrm{X}'_x = \mathrm{X}_x + \mathrm{X}^*_x\,, && \mathrm{X}'_{xy} = \mathrm{X}_{xy} + \mathrm{X}^*_{yx}\,, \\ &\mathrm{X}'_{yx} = \mathrm{X}_{yx} + \mathrm{X}^*_{xy}\,, && \mathrm{X}'_y = \mathrm{X}_y + \mathrm{X}^*_y\,, \\ &\delta'_{cx} = \delta_{cx} - \delta^*_{cx}\,, && \delta'_{cy} = \delta_{cy} - \delta^*_{cy}\,. \end{aligned} \tag{3.6}$$

Comparing equations (3.1) and (3.5), the relations are drawn between the new displacement parameters u', v', w', φ'_x, φ'_y, ω'_z, and the generalised mean displacements of the section u_m, v_m, w_m, φ_x, φ_y, ω_{zm}, defined in chapter 2. In detail, it must result

$$\begin{aligned} &\frac{dw'}{dz} = \varepsilon_{zm} = \frac{dw_m}{dz}\,, \qquad \frac{d\varphi'_x}{dz} = -\chi_x = \frac{d\varphi_x}{dz}\,, \qquad \frac{d\varphi'_y}{dz} = \chi_y = \frac{d\varphi_y}{dz}\,, \\ &-\varphi'_y + \frac{du'}{dz} = \gamma_{zxm}\left(1+\mathrm{X}'_x\right)+\gamma_{zym}\,\mathrm{X}'_{xy}+\theta\,\delta'_{cy} = -\varphi_y + \frac{du_m}{dz} + \gamma_{zxm}\,\mathrm{X}'_x + \gamma_{zym}\,\mathrm{X}'_{xy} + \theta\,\delta'_{cy}\,, \\ &\varphi'_x + \frac{dv'}{dz} = \gamma_{zxm}\,\mathrm{X}'_{yx}+\gamma_{zym}\left(1+\mathrm{X}'_y\right)-\theta\,\delta'_{cx} = \varphi_x + \frac{dv_m}{dz} + \gamma_{zxm}\,\mathrm{X}'_{yx} + \gamma_{zym}\,\mathrm{X}'_y - \theta\,\delta'_{cx}\,, \\ &\frac{d\omega'_z}{dz} = \theta = \frac{d\omega_{zm}}{dz}\,, \end{aligned}$$

and hence it is possible to assume, taking into account the cylinder equilibrium equations (2.13) and (2.14),

$$\begin{aligned} &w' = w_m\,, \qquad \varphi'_x = \varphi_x\,, \qquad \varphi'_y = \varphi_y\,, \qquad \omega'_z = \omega_{zm}\,, \\ &u' = u_m + \omega_z\,\delta'_{cy} - \frac{2\left(1+\nu\right)}{A}\left(\chi_y\,\mathrm{X}'_x\,J_y + \chi_x\,\mathrm{X}'_{xy}\,J_x\right), \\ &v' = v_m - \omega_z\,\delta'_{cx} - \frac{2\left(1+\nu\right)}{A}\left(\chi_y\,\mathrm{X}'_{yx}\,J_y + \chi_x\,\mathrm{X}'_y\,J_x\right). \end{aligned} \tag{3.7}$$

These relations represent the kinematic compatibility equations of the cylinder in terms of the new cross-section displacements. Examining these equations, we can observe that the axis inclination in one plane is not only caused by the shear deformation in the plane itself, but also by the one experienced in the orthogonal plane, and by the torsional deformation. Specifically, this latter one generates displacements of the barycentric axis, resulting from a torsional motion of the cylinder around a rotation axis located at point C′, whose coordinates are δ'_{cx}, δ'_{cy}, and which turns out to be, by definition, the torsional centre of the cross-section. In addition, we can observe that the terms X' featured in equations (3.7) may be regarded as the shear deformability coefficients of the section, depending on the shape of the section itself; in particular, the entities $1+X'_y$, $1+X'_x$ are usually denoted as *shear deformability factors* in *y* and *x* directions. Evidently, such factors assumed unit value within the mechanical representation described in chapter 2, based on the average displacements and rotations of the section.

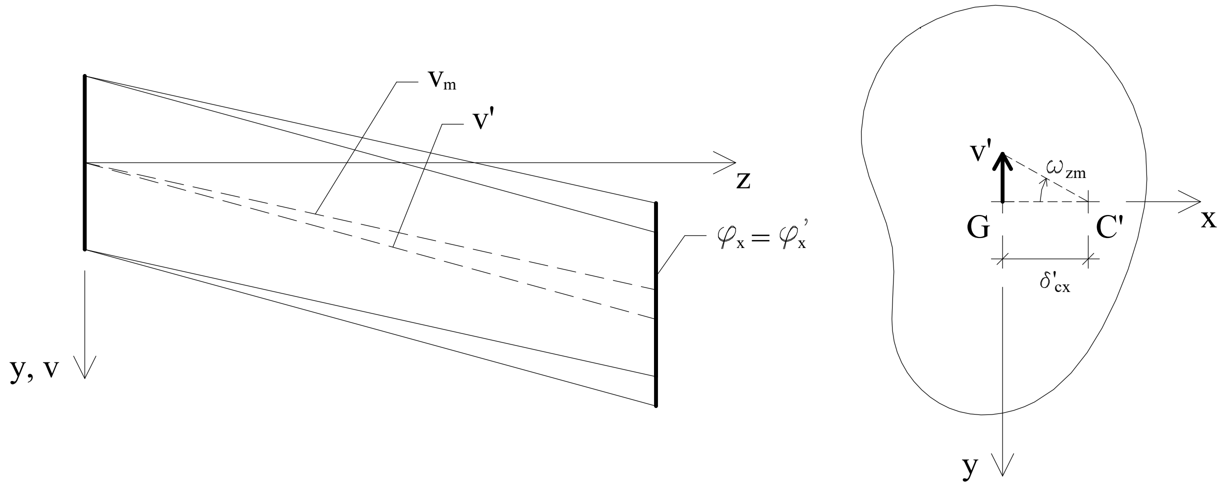

Fig. 3. Axis inclination and torsional center.

From equations (3.7) and from the kinematic compatibility relations highlighted in chapter 2, the torsional centre turns out to be the point around which the section must rotate, in the case of a cylinder exclusively subjected to torsional deformation, in order to find null flexural rotations of the sections around transversal axes.

It must be remarked that equation (3.4) proves to be valid both in terms of actual work and virtual work, while equation (3.5) applies exactly only in terms of actual work.

3.2. Shear centre and torsional centre

Consider two systems of internal forces and deformations, consistent with equations (2.20), referring to the internal forces of shear and torsional moment

$$T_{y.1}\,; \quad \gamma_{yzm.1} = \frac{T_{y.1}}{GA}, \quad \theta_1 = -\frac{q\delta_{tx}}{GJ_0} T_{y.1}$$
$$M_{z.2}\,; \quad \theta_2 = \frac{q}{GJ_0} M_{z.2} \qquad (3.8)$$

The application of the Betti reciprocity theorem, by means of equation (3.4), yields the following expressions for the mutual deformation work

$$\frac{dL_{1-2}(z)}{dz} = -T_{y.1}\,\theta_2\,\delta_{cx} = M_{z.2}\left[\theta_1 + \frac{qA}{J_0}\gamma_{zym.1}\,\delta^*_{cx}\right];$$

it therefore results, substituting the expressions given by equation (3.8),

$$\delta_{cx} = \delta_{tx} - \delta^*_{cx}\,. \qquad (3.9)$$

In the same way, with respect to the internal forces T_y and M_z, it follows

$$\delta_{cy} = \delta_{ty} - \delta^*_{cy}\,. \qquad (3.10)$$

Eventually, applying the theorem to both shear internal forces T_y and T_x, the identity ensues between the joint deformability factors

$$\mathrm{X}'_{xy} = \mathrm{X}'_{yx}\,. \qquad (3.11)$$

The same results may be obtained elaborating the expressions of the abovementioned quantities; for example, in order to prove equation (3.9), consider

$$\delta_{cx} = -\frac{1}{J_x}\int_A \psi_M\, y\, dA\,,$$
$$\delta_{tx} = -\frac{1}{A}\int_A \left(\frac{\partial \psi_{Ty}}{\partial x} y - \frac{\partial \psi_{Ty}}{\partial y} x\right) dA - \frac{1}{A J_x}\int_A (f\,x - g\,y)\, dA\,,$$
$$\delta^*_{cx} = -\frac{1}{A J_x}\int_A \left[\frac{\partial \psi_M}{\partial x} g + \frac{\partial \psi_M}{\partial y} f + (f\,x - g\,y)\right] dA\,.$$

On the basis of such expressions, equation (3.9) becomes

$$-\frac{1}{J_x}\int_A \psi_M\, y\, dA = \frac{1}{A}\int_A \left[-\frac{\partial \psi_{Ty}}{\partial x} y + \frac{\partial \psi_{Ty}}{\partial y} x + \frac{1}{J_x}\left(\frac{\partial \psi_M}{\partial x} g + \frac{\partial \psi_M}{\partial y} f \right)\right] dA. \qquad (3.12)$$

In order to prove the above equation, the integrand function at the second member is put in the form

$$\begin{aligned}
&-\frac{\partial \psi_{Ty}}{\partial x} y + \frac{\partial \psi_{Ty}}{\partial y} x + \frac{1}{J_x}\left(\frac{\partial \psi_M}{\partial x} g + \frac{\partial \psi_M}{\partial y} f \right) = \\
&= -\left(\nabla \psi_{Ty}\right)^{\mathbf{T}} \begin{bmatrix} y \\ -x \end{bmatrix} + \frac{1}{J_x}\left(\nabla \psi_M\right)^{\mathbf{T}} \begin{bmatrix} g \\ f - J_x \end{bmatrix} + \frac{\partial \psi_M}{\partial y} = \\
&= -\nabla^{\mathbf{T}}\left(\psi_{Ty} \begin{bmatrix} y \\ -x \end{bmatrix} \right) + \frac{1}{J_x}\left[\nabla^{\mathbf{T}}\left(\psi_M \begin{bmatrix} g \\ f - J_x \end{bmatrix} \right) - \psi_M \nabla^{\mathbf{T}} \begin{bmatrix} g \\ f - J_x \end{bmatrix} \right] + \frac{\partial \psi_M}{\partial y},
\end{aligned}$$

therefore, the integral itself results

$$-\frac{1}{A}\int_A \nabla^{\mathbf{T}}\left(\psi_{Ty} \begin{bmatrix} y \\ -x \end{bmatrix} \right) dA + \frac{1}{A\,J_x}\int_A \left[\nabla^{\mathbf{T}}\left(\psi_M \begin{bmatrix} g \\ f - J_x \end{bmatrix} \right) - \psi_M \nabla^{\mathbf{T}} \begin{bmatrix} g \\ f - J_x \end{bmatrix} \right] dA, \quad (3.13)$$

since the average value over the section of the functions ψ_M partial derivatives equals zero.

Applying the Gauss theorem, and taking into account equations (2.17), we obtain

$$\begin{aligned}
&\int_A \nabla^{\mathbf{T}}\left(\psi_{Ty} \begin{bmatrix} y \\ -x \end{bmatrix} \right) dA = \int_\Gamma \psi_{Ty}\, \mathbf{n}^{\mathbf{T}} \begin{bmatrix} y \\ -x \end{bmatrix} ds = \int_\Gamma \psi_{Ty}\, \mathbf{n}^{\mathbf{T}}\, \nabla \psi_M\, ds = \\
&= \int_A \nabla^{\mathbf{T}}\left(\psi_{Ty}\, \nabla \psi_M \right) dA = \int_A \left[\left(\nabla \psi_{Ty}\right)^{\mathbf{T}} \nabla \psi_M + \psi_{Ty}\, \Delta \psi_M \right] dA.
\end{aligned} \qquad (3.14)$$

In the same way, it follows that

$$\begin{aligned}
&\frac{1}{J_x}\int_A \nabla^{\mathbf{T}}\left(\psi_M \begin{bmatrix} g \\ f - J_x \end{bmatrix} \right) dA = \frac{1}{J_x}\int_\Gamma \psi_M\, \mathbf{n}^{\mathbf{T}} \begin{bmatrix} g \\ f - J_x \end{bmatrix} ds = \int_\Gamma \psi_M\, \mathbf{n}^{\mathbf{T}}\, \nabla \psi_{Ty}\, ds = \\
&= \int_A \nabla^{\mathbf{T}}\left(\psi_M\, \nabla \psi_{Ty} \right) dA = \int_A \left[\left(\nabla \psi_M\right)^{\mathbf{T}} \nabla \psi_{Ty} + \psi_M\, \Delta \psi_{Ty} \right] dA.
\end{aligned} \qquad (3.15)$$

Introducing the results of equations (3.14) and (3.15) into expression (3.13), this latter transforms into

$$-\frac{1}{A}\int_A \psi_{Ty}\Delta\psi_M \, dA + \frac{1}{A}\int_A \psi_M \Delta\psi_{Ty} \, dA - \frac{1}{A J_x}\int_A \psi_M \nabla^{\mathrm{T}} \begin{bmatrix} g \\ f - J_x \end{bmatrix} dA. \qquad (3.16)$$

Given that, on the basis of equations (2.16) and (2.17), the following relations are satisfied

$$\Delta\psi_M = 0, \qquad \Delta\psi_{Ty} = -\frac{A}{(1+\nu)J_x} y, \qquad \nabla^{\mathrm{T}} \begin{bmatrix} g \\ f - J_x \end{bmatrix} = \frac{\nu A}{1+\nu} y,$$

expression (3.16) then becomes

$$-\frac{1}{(1+\nu)J_x}\int_A \psi_M \, y \, dA - \frac{\nu}{(1+\nu)J_x}\int_A \psi_M \, y \, dA = -\frac{1}{J_x}\int_A \psi_M \, y \, dA.$$

Since expression (3.16) derives from the second member of equation (3.9), the equation itself is demonstrated. In a similar way, equations (3.10) and (3.11) can be proved.

On the basis of relations (3.9) and (3.10), the coordinates of the torsional centre, defined by equations (3.6), may be written in the form

$$\delta'_{cx} = 2\delta_{cx} - \delta_{tx} , \qquad \delta'_{cy} = 2\delta_{cy} - \delta_{ty} .$$

Generally, the quantities δ^*_{cx} e δ^*_{cy} may be regarded as negligible; therefore, from equations (3.9) and (3.10), we can argue that the torsional centre and the shear centre are neighbouring.

3.3. The internal deformation work

At this point of the process, we set the goal to determine the expressions of the shear deformability factors defined by equations (3.6), on the basis of the internal deformation work, evaluated through the application of the Clapeyron theorem. On this purpose, and for the sake of simplicity, the cylinder is supposed to be subjected only to bending and shear deformations along the *y*-*z* plane; in other words, we assign

$$\varepsilon_{zm} = 0 \quad , \quad \theta = 0 \quad , \quad \gamma_{zxm} = 0 \quad , \quad \chi_y = 0 \, .$$

In such conditions, the local strains defined by equations (2.19) reduces to

$$\begin{aligned} &\varepsilon_z = -\chi_x \, y \, , \\ &\gamma_{zx} = \gamma_{zym} \left[\frac{\partial \psi_{Ty}}{\partial x} - \frac{g(x,y)}{J_x} \right] , \\ &\gamma_{zy} = \gamma_{zym} \left[1 + \frac{\partial \psi_{Ty}}{\partial y} - \frac{f(x,y)}{J_x} \right] . \end{aligned} \qquad (3.17)$$

Referring again to the cylinder elementary portion with length *dz*, the external work may be described, by means of equation (3.5), as it follows

$$\frac{dL_e}{dz} = -M_x \, \chi_x + T_y \, \gamma_{zym} \left(1 + \mathrm{X}'_y \right) , \qquad (3.18)$$

The internal work, on the other hand, is evaluable as

$$dL_i = \int_A \left(\sigma_z \varepsilon_z + \tau_{zx} \gamma_{zx} + \tau_{zy} \gamma_{zy} \right) dA \, dz \, , \qquad (3.19)$$

therefore, substituting the expressions of stresses and strains given by equation (3.17) into equation (3.19), the internal work may be written as

$$\begin{aligned} dL_i = \Bigg[-M_x \, \chi_x + T_y \, \gamma_{zym} + \frac{T_y \, \gamma_{zym}}{A} \int_A \Bigg[\left(\frac{\partial \psi_{Ty}}{\partial x} \right)^2 + \left(\frac{\partial \psi_{Ty}}{\partial y} \right)^2 + \\ - \frac{2}{J_x} \left[\frac{\partial \psi_{Ty}}{\partial x} g + \frac{\partial \psi_{Ty}}{\partial y} f \right] + \frac{f^2 + g^2}{J_x^2} \Bigg] dA \Bigg] dz \, . \end{aligned}$$

The application of the Clapeyron theorem, with regards to the cylinder differential portion, results in imposing that

$$\frac{dL_e}{dz}dz = dL_i\,,$$

and therefore, the following relation is derived

$$\mathrm{X}'_y = \frac{1}{A}\int_A \left[\left(\frac{\partial \psi_{Ty}}{\partial x}\right)^2 + \left(\frac{\partial \psi_{Ty}}{\partial y}\right)^2 - \frac{2}{J_x}\left[\frac{\partial \psi_{Ty}}{\partial x}g + \frac{\partial \psi_{Ty}}{\partial y}f\right] + \frac{f^2+g^2}{J_x^{\,2}}\right]dA\,, \qquad (3.20)$$

which consists in the expression of the shear deformability factor in y direction, attained by means of the internal work.

Actually, from equations (3.3) and (3.6), the following relations are also verified

$$\mathrm{X}'_y = \mathrm{X}_y + \mathrm{X}^*_y = \frac{1}{J_x}\int_A \psi_{Ty}\, y\, dA - \frac{1}{A\,J_x}\int_A \left[\frac{\partial \psi_{Ty}}{\partial x}g + \frac{\partial \psi_{Ty}}{\partial y}f - \frac{f^2+g^2}{J_x}\right]dA\,, \qquad (3.21)$$

therefore, comparing equations (3.20) and (3.21), it results that

$$\frac{1}{J_x}\int_A \psi_{Ty}\, y\, dA = \frac{1}{A}\int_A \left[\left(\frac{\partial \psi_{Ty}}{\partial x}\right)^2 + \left(\frac{\partial \psi_{Ty}}{\partial y}\right)^2 - \frac{1}{J_x}\left[\frac{\partial \psi_{Ty}}{\partial x}g + \frac{\partial \psi_{Ty}}{\partial y}f\right]\right]dA\,. \qquad (3.22)$$

Such relation may be proved rewriting the integrand function at the second member in the following form

$$\left(\frac{\partial \psi_{Ty}}{\partial x}\right)^2 + \left(\frac{\partial \psi_{Ty}}{\partial y}\right)^2 - \frac{1}{J_x}\left(\frac{\partial \psi_{Ty}}{\partial x}g + \frac{\partial \psi_{Ty}}{\partial y}f\right) =$$
$$= \left(\nabla \psi_{Ty}\right)^{\mathrm{T}} \nabla \psi_{Ty} - \frac{1}{J_x}\left(\nabla \psi_{Ty}\right)^{\mathrm{T}} \begin{bmatrix} g \\ f - J_x \end{bmatrix} - \frac{\partial \psi_{Ty}}{\partial y} =$$
$$= \nabla^{\mathrm{T}}\left(\psi_{Ty}\,\nabla \psi_{Ty}\right) - \psi_{Ty}\,\Delta \psi_{Ty} - \frac{1}{J_x}\left[\nabla^{\mathrm{T}}\left(\psi_{Ty}\begin{bmatrix} g \\ f - J_x \end{bmatrix}\right) - \psi_{Ty}\,\nabla^{\mathrm{T}}\begin{bmatrix} g \\ f - J_x \end{bmatrix}\right] - \frac{\partial \psi_{Ty}}{\partial y}.$$

Applying the Gauss theorem and taking into account equations (2.17), we obtain that

$$\int_A \nabla^{\mathrm{T}} \left(\psi_{Ty} \nabla \psi_{Ty} \right) dA = \int_\Gamma \psi_{Ty} \, \mathbf{n}^{\mathrm{T}} \nabla \psi_{Ty} \, ds = \frac{1}{J_x} \int_\Gamma \psi_{Ty} \, \mathbf{n}^{\mathrm{T}} \begin{bmatrix} g \\ f - J_x \end{bmatrix} ds =$$

$$= \frac{1}{J_x} \int_A \nabla^{\mathrm{T}} \left(\psi_{Ty} \begin{bmatrix} g \\ f - J_x \end{bmatrix} \right) dA \, ,$$

In addition, the second member of equation (3.22) assumes the following expression

$$-\frac{1}{A} \int_A \psi_{Ty} \, \Delta \psi_{Ty} \, dA + \frac{1}{A J_x} \int_A \psi_{Ty} \, \nabla^{\mathrm{T}} \begin{bmatrix} g \\ f - J_x \end{bmatrix} dA \, , \qquad (3.23)$$

since the partial derivatives of functions ψ_{Ty} have zero mean values throughout the section. Actually, equations (2.16) and (2.17) yield as well

$$\Delta \psi_{Ty} = -\frac{A}{(1+\nu) J_x} y \, , \qquad \nabla^{\mathrm{T}} \begin{bmatrix} g \\ f - J_x \end{bmatrix} = \frac{\nu A}{1+\nu} y \, .$$

Substituting such expressions in equation (3.23), we obtain

$$\frac{1}{(1+\nu) J_x} \int_A \psi_{Ty} \, y \, dA + \frac{\nu}{(1+\nu) J_x} \int_A \psi_{Ty} \, y \, dA = \frac{1}{J_x} \int_A \psi_{Ty} \, y \, dA \, ,$$

and therefore relation (3.22) is demonstrated. Similar assessments apply with regards to the shear deformability factor in x direction, and to the joint deformability factors with regards to x and y axes.

4. CONCLUSIONS

In the present study, the displacements field experienced by the Saint Venant cylinder has been determined referring from the beginning to the average displacements and rotations of the cross-section. Among these latter ones, the flexural rotations around the two transversal axes have been defined as the integral average value of the corresponding local rotations.

On the basis of such methodology, the expressions describing the cylinder motion contain the section deformation parameters, besides the generalised mean displacements.

The kinematic relations between such deformation parameters and the derivatives of the mean displacements highlight that, in such a conceived model, the cylinder deformation status is characterised by unit shear factors and a torsional centre coinciding with the section centroid. Anyway, the typical solution methods for this specific problem usually involve a different set of characteristic displacements of the cross-section, attained through assessments of energetic kind. Basically, a displacements system is considered for which the work produced by the internal forces is equal to the one effected by the stresses over the whole section, due to the local displacements.

Adopting this kinematic model, relations between shear deformations, flexural rotations and axis inclinations becomes less direct; in effect, they turn out to involve the shear deformability factors and the torsional centre, which is not coincident anymore with the section centroid, but rather close to the shear centre [1]. In addition, it may be noticed that, unless special symmetry conditions are given, shear deformations in a main plane result in inclination of the cylinder axis in the orthogonal plane.

Anyway, arbitrariness only concerns the problem solution in terms of displacements; in effect, local strains and stresses distributions are univocally determined, since their expressions are not affected by the definition adopted for the generalised section displacements.

Relating these two sets of displacements parameters, the expressions of the shear deformability factors and the coordinates of the torsional centre are derived. The former ones result formally different from the expressions obtained through the application of the Clapeyron theorem to the infinitesimal transversal portion of the solid; anyway, these expressions are totally equivalent if the theorem is rigorously applied, i.e., if the external work is evaluated not neglecting the contribution due to the displacements system having null average derivatives over the section.

5. REFERENCES

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